# *On the Part of the Parallactic Inequalities in the Moon's Motion which is a Function of the Mean Motions of the Sun and Moon.*

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In Vol. I of the *American Journal of Mathematics*,* Mr. G. W. Hill, taking into account only inequalities depending on the mean motions of the sun and moon, has shown that these inequalities are able to be determined to a high degree of accuracy by using moving rectangular axes, one of which passes through the mean place of the sun. This paper is an adaptation of his method so as to include that class of inequalities which depends also on the ratio of the solar and lunar distances, and in particular, the principal part of the Parallactic Inequality. Owing to the use which has been made of this latter in obtaining the parallax of the sun, it becomes of importance that its coefficient should be accurately known. Delaunay's expressions† are deficient in this respect, and we have no means of knowing how near Hansen's numerical value is to the truth. The inequalities are obtained below in an algebraical and numerical form, the latter giving their coefficients in longitude and parallax true to about one thousandth of a second of arc.

## I.

### *Transformation of the Equations of Motion.*

The inequalities from purely circular motion which depend only on the lunar eccentricity and mean motions of the sun and moon only, are given by the equations

$$\left.\begin{aligned}\frac{d^2x}{dt^2} - 2n'\frac{dy}{dt} + \left(\frac{\mu}{r^3} - 3n'^2\right)x &= 0,\\ \frac{d^2y}{dt^2} + 2n'\frac{dx}{dt} + \frac{\mu}{r^3}\,y &= 0,\end{aligned}\right\} \tag{1}$$

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* Researches in the Lunar Theory, pp. 5, 129, 245. † Mém. Fr. Acad. Sc., Vol. XXIX, p. 847.

where $n'$, $\mu$ have their usual significations, the axes being rectangular and revolving with uniform angular velocity $n'$, and that of $x$ passing through the mean sun. The earth is supposed to be moving in a circle round the sun with uniform angular velocity $n'$. The above equation is that which would be obtained from the consideration of a disturbing body of infinitely great mass, and at an infinite distance moving round the earth in a circle with finite uniform angular velocity $n'$, where $n'^2 = \text{mass} \div (\text{dist.})^3$. In order to get the inequalities free from the lunar eccentricity, Mr. Hill gets a particular solution of the above equations. When now we wish to include inequalities dependent on the distance of the sun, we can no longer suppose it at an infinite distance and of infinite mass. We must include the part of the disturbing function due to this cause which has been omitted. It will then be seen that we can obtain all the inequalities which a disturbing body moving round the earth with constant angular velocity in the moon's orbit produces, when the moon's undisturbed orbit is taken to be circular.

Hence we have, instead of zero on the right-hand sides of equations (1), to put $\frac{d\Omega_1}{dx}$ and $\frac{d\Omega_1}{dy}$ respectively, where

$$\Omega_1 = \frac{n'^2}{a'}\left[x^3 - \frac{3}{2}xy^2\right] + \frac{n'^2}{a'^2}\left[x^4 - 3x^2y^2 + \frac{3}{8}y^4\right] + \frac{n'^2}{a'^3}\left[x^5 - 5x^3y^2 + \frac{15}{8}xy^4\right] + \dots.$$

Multiplying the equations thus formed by $\frac{dx}{dt}$, $\frac{dy}{dt}$, adding and integrating the result, we get the Jacobian equation

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 - \frac{2\mu}{r} - 3n'^2x^2 = 2\Omega_1 - 2C.$$

Transforming to new coordinates $u$, $s$, where

$$u = x + y\sqrt{-1},$$
$$s = x - y\sqrt{-1},$$

our equations of motion and the Jacobian integral become respectively

$$\left.\begin{aligned}
&\frac{d^2u}{dt^2} + 2n'\sqrt{-1}\,.\frac{du}{dt} + \frac{\mu}{(us)^{\frac{3}{2}}}u = \frac{3}{2}n'^2(u+s) + 2\frac{d\Omega_1}{ds},\\
&\frac{d^2s}{dt^2} - 2n'\sqrt{-1}\,.\frac{ds}{dt} + \frac{\mu}{(us)^{\frac{3}{2}}}\,.\,s = \frac{3}{2}n'^2(u+s) + 2\frac{d\Omega_1}{du},\\
&\frac{du}{dt}\cdot\frac{ds}{dt} = \frac{2\mu}{(us)^{\frac{1}{2}}} + \frac{3}{4}n'^2(u+s)^2 + 2\Omega_1 - 2C,
\end{aligned}\right\} \qquad (2)$$

and $\Omega_1$ takes the form

$$\Omega_1 = \frac{n'^2}{a'}\left[\frac{5}{16}(u^3+s^3)+\frac{3}{16}us(u+s)\right]$$
$$+\frac{n'^2}{a'^2}\left[\frac{35}{128}(u^4+s^4)+\frac{5}{32}us(u^2+s^2)+\frac{9}{64}u^2s^2\right]$$
$$+\frac{n'^2}{a'^3}\left[\frac{63}{256}(u^5+s^5)+\frac{35}{256}us(u^3+s^3)+\frac{15}{128}u^2s^2(u+s)\right]$$
$$+\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

$\Omega_1$, expressed in terms of $u$, $s$, is obtained directly by expanding $\dfrac{n'^2a'^3}{\rho}$, where $\rho$ is the distance between the sun and moon and is equal in this case to

$$\sqrt{(a'^2 - 2a'r\cos\theta + r^2)}.$$

For since $r^2 = us$ and $2r\cos\theta = 2x = u + s$, we have $2r^n\cos n\theta = u^n + s^n$, and therefore

$$\frac{n'^2a'^3}{\rho} = n'^2a'^2\left(1 + \frac{r}{a'}P_1 + \frac{r^2}{a'^2}P_2 + \ldots.\right).$$

$P_n$ is the zonal harmonic of degree $n$, and is expressible in terms of

$$\cos n\theta,\ \cos(n-2)\theta\ \ldots.$$

Hence we have

$$\Omega_1 = r^3\frac{n'^2}{a'}P_3 + r^4\frac{n'^2}{a'^3}P_4 + \ldots.,$$

where

$$r^nP_n = \frac{1.3\ldots.(2n-1)}{2.4\ldots.2n}(u^n+s^n) + \frac{1.3\ldots.(2n-3)}{2.4\ldots.(2n-2)}\cdot\frac{1}{2}us(u^{n-2}+s^{n-2})+\ldots.$$

## II.

### *Solution of the Equations.*

In order to solve our equations for the particular class of inequalities which we wish to get, take the particular integrals

$$x = \Sigma A_i \cos i\nu(t-t_0),$$
$$y = \Sigma A_i \sin i\nu(t-t_0),$$

$i$ having positive integral values from zero to infinity, and $\nu = n - n'$. Put

$$A_i = a_{i-1} + a_{-i-1},\quad B_i = a_{i-1} - a_{-i-1},$$

we obtain

$$x = \Sigma a_{i-1}\cos i\nu(t-t_0),$$
$$y = \Sigma a_{i-1}\sin i\nu(t-t_0),$$

where the summation is now extended to negative values of $i$. Transforming to the complex variables $u$, $s$, and putting $e^{\nu(t-t_0)\sqrt{-1}}=\zeta$, we have

$$u=\Sigma a_{i-1}\zeta^i,$$
$$s=\Sigma a_{-i-1}\zeta^i.$$

Also putting

$$m=\frac{n'}{\nu}\,,\quad \varkappa=\frac{\mu}{\nu^2}\,,\quad \zeta\frac{d}{d\zeta}=-\frac{\sqrt{-1}}{\nu}\cdot\frac{d}{dt}=D,$$

the equations (2) may be written

$$\left[D^2+2mD-\frac{\varkappa}{(us)^{\frac{3}{2}}}\right]u=-\frac{3}{2}m^2(u+s)-m^2.\frac{2}{n'^2}\cdot\frac{d\Omega_1}{ds}\,,$$
$$\left[D^2-2mD-\frac{\varkappa}{(us)^{\frac{3}{2}}}\right]s=-\frac{3}{2}m^2(u+s)-m^2.\frac{2}{n'^2}\cdot\frac{d\Omega_1}{du}\,,$$
$$Du.Ds+\frac{2\varkappa}{(us)^{\frac{1}{2}}}=-\frac{3}{4}m^2(u+s)^2-m^2.\frac{2}{n'^2}.\Omega_1+C'.$$

Multiply the first of these equations by $s$, the second by $u$, and add to the last; also with the same multipliers subtract the first from the second, the resulting equations will be

$$\left.\begin{aligned}&D^2(us)-DuDs-2m(uDs-sDu)+\frac{9}{4}m^2(u+s)^2\\&\qquad\qquad=-m^2.\frac{2}{n'^2}\left(u\frac{d\Omega_1}{du}+s\frac{d\Omega_1}{ds}+\Omega_1\right)+C',\\&D(uDs-sDu-2mus)+\frac{3}{2}m^2(u^2-s^2)=-m^2.\frac{2}{n'^2}\left(u\frac{d\Omega_1}{du}-s\frac{d\Omega_1}{ds}\right)\cdot\end{aligned}\right\}(3)$$

The constant $\varkappa$ has vanished and will have to be determined from our previous equations in terms of the new constants introduced in the Particular Solution (Hill, p. 132).

Substituting in (3) the values of $u$, $s$, which are

$$u=\Sigma_i a_{i-1}\zeta^i,\qquad s=\Sigma_i a_{-i-1}\zeta^i,$$

or, what is the same thing,

$$u=\Sigma_i a_i\zeta^{i+1},\qquad s=\Sigma_i a_{-i}\zeta^{i-1},$$

so that

$$us=\Sigma_i\Sigma_j a_j a_{j-i}\zeta^i,\qquad uDs-sDu=\Sigma_i\Sigma_j(i-2j-2)\,a_j a_{j-i}\zeta^i,\ \text{etc.},$$

and equating the coefficient of $\zeta^i$ to zero (except when $i = 0$), we obtain

$$\left.\begin{aligned}
&\Sigma_j \left[i^2 - (j+1)(i-j-1) - 2m(i-2j-2) + \frac{9}{2}m^2\right] a_j a_{j-i} \\
&\qquad + \frac{9}{4}m^2 \Sigma_j \left[a_{j-1}a_{i-j-1} + a_{-j-1}a_{-i+j-1}\right] = -m^2 L_i, \\
&\Sigma_j \left[i(i-2j-2) - 2mi\right] a_j a_{j-i} \\
&\qquad + \frac{3}{2}m^2 \Sigma_j \left[a_{j-1}a_{i-j-1} - a_{-j-1}a_{-i+j-1}\right] = -m^2 M_i,
\end{aligned}\right\} \quad (4)$$

where $-m^2L_i$, $-m^2M_i$ are the coefficients of $\zeta^i$ on the right-hand sides of equations (3). Multiply these equations by 2 and 3 respectively, and take the sum and difference, we get

$$\begin{aligned}
&\Sigma_j \left[5i^2 - 8(j+1)i + 2(j+1)^2 - 2m(5i-4j-4) + 9m^2\right] a_j a_{-i+j} \\
&\qquad + 9m^2 \Sigma_j a_{j-1}a_{i-j-1} = -m^2(2L_i + 3M_i), \\
&\Sigma_j \left[-i^2 + 4(j+1)i + 2(j+1)^2 + 2m(i+4j+4) + 9m^2\right] a_j a_{-i+j} \\
&\qquad + 9m^2 \Sigma_j a_{-j-1}a_{-i+j-1} = -m^2(2L_i - 3M_i).
\end{aligned}$$

The terms of lowest order in these equations are $a_0 a_i$ and $a_0 a_{-i}$. In order to separate out these, multiply the last pair of equations by

$$\begin{gathered}
-i^2 + 4i + 2 + 2m(i+4) + 9m^2, \\
-5i^2 + 8i - 2 + 2m(5i-4) - 9m^2,
\end{gathered}$$

respectively, add the products and divide the whole by

$$12i^2\left[2(i^2-1) - 4m + m^2\right].$$

In the resulting equation the term $a_0 a_{-i}$ will have vanished and the coefficient of $a_0 a_i$ will be $-1$. The equation is

$$\left.\begin{aligned}
&\Sigma_j \left\{[i,j]\, a_j a_{-i+j} + [i]\, a_{j-1}a_{i-j-1} + (i)\, a_{-j-1}a_{-i+j-1}\right\} \\
&\qquad = -\frac{1}{9}\left[2L_i\{[i] + (i)\} + 3M_i\{[i] - (i)\}\right],
\end{aligned}\right\} \quad (5)$$

where
$$\begin{aligned}
[i,j] &= -\frac{j}{i} \cdot \frac{(i-2)j + i^2 + 2i - 2 - 2(j-i+2)m + m^2}{2(i^2-1) - 4m + m^2}, \\
[i] &= -\frac{3m^2}{4i^2} \; \frac{i^2 - 4i - 2 - 2(i+4)m - 9m^2}{2(i^2-1) - 4m + m^2}, \\
(i) &= -\frac{3m^2}{4i^2} \; \frac{5i^2 - 8i + 2 - 2(5i-4)m + 9m^2}{2(i^2-1) - 4m + m^2},
\end{aligned}$$

and therefore, as they should be,

$$[i, i] = -1 \qquad [i, 0] = 0.$$

The equation (5) corresponds to that obtained by Mr. Hill on p. 135 of his memoir referred to. It may be shown that if we put zero instead of the function on the right-hand side of equation (5), every $a$ with an odd suffix will vanish, and with one or two changes in notation it becomes the same as his.

The expressions $L_i$, $M_i$ must now be obtained. Since $\Omega_1$ is formed of homogeneous functions of $u$, $s$ of the 3^d^, 4^th^, .... degrees, we have

$$\Omega_1 + u\frac{d\Omega_1}{du} + s\frac{d\Omega_1}{ds} = 4 \cdot \frac{n'^2}{a'}\left[\frac{5}{16}(u^3 + s^3) + \frac{3}{16}us(u+s)\right]$$
$$+ 5 \cdot \frac{n'^2}{a'^2}\left[\frac{35}{128}(u^4 + s^4) + \frac{5}{32}us(u^2 + s^2) + \frac{9}{64}u^2s^2\right]$$
$$+ 6 \cdot \frac{n'^2}{a'^3}\left[\frac{63}{256}(u^5 + s^5) + \frac{35}{256}us(u^3 + s^3) + \frac{15}{128}u^2s^2(u+s)\right] + \ldots\ldots,$$

$$u\frac{d\Omega_1}{du} - s\frac{d\Omega_1}{ds} = \frac{n'^2}{a'}\left[\frac{15}{16}(u^3 - s^3) + \frac{3}{16}us(u-s)\right]$$
$$+ \frac{n'^2}{a'^2}\left[\frac{35}{32}(u^4 - s^4) + \frac{5}{16}us(u^2 - s^2)\right]$$
$$+ \frac{n'^2}{a'^3}\left[\frac{315}{256}(u^5 - s^5) + \frac{105}{256}us(u^3 - s^3) + \frac{15}{128}u^2s^2(u-s)\right] + \ldots\ldots,$$

and $\frac{n'^2}{2}L_i$, $\frac{n'^2}{2}M_i$ being the coefficients in these when for $u$, $s$ are substituted their values. Hence

$$-\frac{1}{9}\left[2L_i\{[i] + (i)\} + 3M_i\{[i] - (i)\}\right]$$
$$= -\frac{1}{a'}\left[A_i(u^3)_i + A'_i(s^3)_i + B_i(u^2s)_i + B'_i(us^2)_i\right]$$
$$-\frac{1}{a'^2}\left[C_i(u^4)_i + C'_i(s^4)_i + D_i(u^3s)_i + D'_i(us^3)_i + E_i(u^2s^2)_i\right]$$
$$-\frac{1}{a'^3}\left[F_i(u^5)_i + F'_i(s^5)_i + G_i(u^4s)_i + G'_i(us^4)_i + H_i(u^3s^2)_i + H'_i(u^2s^3)_i\right],$$
$$- \ldots\ldots\ldots\ldots$$

where $(u^3)_i$, $(s^3)_i$, etc., denote the coefficients of $\zeta^i$ in $u^3$, $s^3$, etc. $A_i$, $A'_i$, etc. are definite functions of $m$ of the order $m^2$ at least. Their values are

$$A_i = \frac{5}{72}\{17[i] - (i)\}, \qquad E_i = \frac{5}{16}\{[i] + (i)\},$$

$$B_i = \frac{1}{24}\{11[i] + 5(i)\}, \qquad F_i = \frac{21}{128}\{9[i] - (i)\},$$

$$C_i = \frac{35}{288}\{11[i] - (i)\}, \qquad G_i = \frac{35}{384}\{7[i] + (i)\},$$

$$D_i = \frac{5}{36}\{4[i] + (i)\}, \qquad H_i = \frac{5}{64}\{5[i] + 3(i)\}.$$

$A'_i$, $B'_i$, $C'_i$, $D'_i$, $F'_i$, $G'_i$, $H'_i$ are got by merely interchanging $[i]$ and $(i)$ in the expressions for the corresponding undashed letters. Hence

$$A_i = -\frac{5m^2}{16i^2}\cdot\frac{2i^2 - 10i - 6 - 4(i+6)m - 27m^2}{2(i^2-1) - 4m + m^2},$$

$$A'_i = -\frac{5m^2}{16i^2}\cdot\frac{14i^2 - 22i + 6 - 4(7i-6)m + 27m^2}{2(i^2-1) - 4m + m^2},$$

$$B_i = -\frac{3m^2}{16i^2}\cdot\frac{6i^2 - 14i - 2 - 4(3i+2)m - 9m^2}{2(i^2-1) - 4m + m^2}.$$

$$B'_i = -\frac{3m^2}{16i^2}\cdot\frac{10i^2 - 18i + 2 - 4(5i-2) + 9m^2}{2(i^2-1) - 4m + m^2},$$

etc.

Also we have

$$L_1 = \frac{2}{a'}\left[\frac{5}{4}\{(u^3)_1 + (s^3)_1\} + \frac{3}{4}\{(u^2s)_1 + (s^2u)_1\}\right] + \ldots\ldots,$$

$$M_1 = \frac{2}{a'}\left[\frac{15}{16}\{(u^3)_1 - (s^3)_1\} + \frac{3}{16}\{(u^2s)_1 - (s^2u)_1\}\right] + \ldots\ldots$$

The coefficients of $\zeta^i$ in $u^3$, $s^3$, etc., are now to be obtained. From the forms of $u$, $s$ we have, if the coefficient of $\zeta^i$ in $u^ps^q$ be denoted by $(u^ps^q)_i$ as before,

$$(u^ps^q)_i = \Sigma\Sigma\ldots\ldots a_ja_ka_l\ldots\ldots(p \text{ factors}) \times a_sa_ta_w\ldots\ldots(q \text{ factors}),$$

for all integral values, positive and negative, of $j$, $k$, $l\ldots\ldots s$, $t$, $w\ldots\ldots$ consistent with the condition

$$(p + j + k + l\ldots\ldots) - (q + s + t + w\ldots\ldots) = i.$$

From this equation we see that

$$(u^p s^q)_i = (s^p u^q)_{-i}.$$

The coefficient of $\zeta^i$ is more easily obtained by taking the coefficient of $\zeta^h$, where

$$h = i - p + q,$$

and therefore

$$(j + k + l + \ldots.) - (s + t + w + \ldots.) = h.$$

## III.

### *The Coefficients $a_1$ and $a_{-1}$.*

We have now obtained all the expressions necessary for the determination of the coefficients $a_i$. But before developing them, some remarks must be made on the equations for $a_1$ and $a_{-1}$.

It will be noticed that every term in equation (5) except the principal one is divided by the expression $2(i^2 - 1) - 4m + m^2$. When $i = \pm 1$, this reduces to $-4m + m^2$, and the order of $a_1$ and $a_{-1}$ is thus lowered by one power of $m$. The consequence of this is a great increase in the difficulty of obtaining their expressions. We have to carry them one order higher to get the same degree of approximation, and when we obtain them by the method of successive approximation, each process, instead of carrying our expressions *two* orders higher, takes them only one; the number of such processes is consequently doubled, and the numerical multipliers of the various powers of $m$ are very much increased in complexity.

A second disadvantage arises from another cause. If we form the expansions of $a_1$ and $a_{-1}$ in ascending powers of $m$ from equation (5), we find large and continually increasing multipliers of the various powers of $m$ which tend to lessen the value of the successive approximations when expressed numerically, to such an extent that the series, instead of proceeding in a progression whose ratio is roughly $m$ or $1/12$, proceeds in a progression with a ratio of about $5m$ or $2/5$. As this ratio seems somewhat regular between the successive powers, I have used the following method to discover and counteract the effect of the slow convergence. In the following, squares and higher powers of $1/a'$ are neglected.

Going back to the equations (4) obtained directly from the equations of motion, put $i = 1$ in them, and write down the coefficients of $a_1$, $a_{-1}$, neglecting terms of the seventh and higher orders. These principal terms then become

$$a_0 a_1 \left[3 + 6m + \frac{9}{2} m^2 + \left(7 + 10m + \frac{9}{2} m^2\right) \frac{a_2}{a_0} + \frac{9}{2} m^2 . \frac{a_{-2}}{a_0}\right]$$
$$+ a_0 a_{-1} \left[1 + 2m + 9m^2 + (1 - 2m + 9m^2) \frac{a_{-2}}{a_0}\right],$$
$$a_0 a_1 \left[- 3 - 2m - (5 + 2m) \frac{a_2}{a_0} + 3m^2 \frac{a_{-2}}{a_0}\right]$$
$$+ a_0 a_{-1} \left[- 1 - 2m + 3m^2 + (1 - 2m + 3m^2) \frac{a_{-2}}{a_0}\right].$$

Now $a_2/a_0$ and $a_{-2}/a_0$ involve $m$ only, and their expressions have been found by Mr. Hill. He gives

$$\frac{a_2}{a_0} = \frac{3}{16} m^2 + \frac{1}{2} m^3 + \frac{7}{12} m^4 + \ldots,$$
$$\frac{a_{-2}}{a_0} = - \frac{19}{16} m^2 - \frac{5}{3} m^3 - \frac{43}{36} m^4 + \ldots$$

Substituting these values and expanding, we have for the principal terms,

$$a_0 a_1 \left[3 + 6m + \frac{93}{16} m^2 + \ldots\right] + a_0 a_{-1} \left[1 + 2m + \frac{125}{16} m^2 + \ldots\right],$$
$$a_0 a_1 \left[- 3 - 2m - \frac{15}{16} m^2 + \ldots\right] + a_0 a_{-1} \left[- 1 - 2m + \frac{29}{16} m^2 + \ldots\right].$$

Denote these expressions by $a_0 a_1 P + a_0 a_{-1} P'$ and $a_0 a_1 Q + a_0 a_{-1} Q'$ respectively.

When we wish to separate out $a_1$ and $a_{-1}$ by the ordinary process of solution, i. e. multiplying the equations by $Q'$, $P'$ and subtracting, also by $Q$, $P$ and subtracting; the principal terms become

$$a_0 a_1 (PQ' - P'Q),$$
$$a_0 a_{-1} (QP' - Q'P).$$

The resulting expressions for $a_1$ and $a_{-1}$ thus found would therefore be both divided by the factor $PQ' - Q'P$. Expanding it in powers of $m$, we find these expressions will be multiplied by

$$- \left(1 - 4m - \frac{37}{8} m^2 - \ldots\right)^{-1} . \frac{1}{4m},$$

or by

$$- \frac{1}{4m} \left(1 + 4m + \frac{165}{8} m^2 + \ldots\right).$$

We thus have an explanation of the slow convergence of the series, and by retaining this divisor, we shall be able to obtain series which will converge much more quickly. By proceeding in the following way, the values of $a_1$ and $a_{-1}$ can be formed to the seventh order in one approximation; and it will also be shown that to this order, our values will be sufficient to obtain the coefficient of this part of the Parallactic Inequality from the expansions, with an error less than two hundredths of a second of arc.

In equations (4) put $i = 1$, we have

$$\Sigma_j a_j a_{j-1}\left[j^2 + j + 1 + 2m(2j+1) + \frac{9}{2}m^2\right] + \Sigma_j \frac{9}{4} m^2 \left[a_{j-1}a_{-j} + a_{-j-1}a_{j-2}\right] = -m^2 L_1,$$

$$\Sigma_j a_j a_{j-1}\left[-2j - 1 - 2m\right] + \Sigma_j \frac{3}{2} m^2 \left[a_{j-1}a_{-j} - a_{-j-1}a_{j-2}\right] = -m^2 M_1.$$

Multiply the second equation by $2m$, add to the first and write out the resulting equation and the second equation, for errors of the sixth order at most in $m$,

$$\left.\begin{aligned}
&a_0a_1\Big[3 + \frac{m^2}{2} + \frac{a_2}{a_0}\Big(7 + \frac{m^2}{2}\Big) + \frac{a_{-2}}{a_0}\Big(\frac{9}{2}m^2 + 6m^3\Big) \\
&\qquad + \frac{a_{-4}}{a_0}\Big(\frac{9}{2}m^2 - 6m^3\Big)\Big] + a_0a_3\Big[\frac{a_2}{a_0}\Big(13 + \frac{m^2}{2}\Big) + \frac{a_4}{a_0}\Big(21 + \frac{m^2}{2}\Big)\Big] \\
&+ a_0a_{-1}\Big[1 + 5m^2 + 6m^3 + \frac{a_{-2}}{a_0}(1 + 5m^2 - 6m^3)\Big] \\
&+ a_0a_{-3}\Big[\frac{9}{2}m^2 - 6m^3 + \frac{a_2}{a_0}\Big(\frac{9}{2}m^2 + 6m^3\Big) + \frac{a_{-2}}{a_0}\Big(3 + \frac{m^2}{2}\Big) \\
&\qquad\qquad + \frac{a_{-4}}{a_0}\Big(7 + \frac{m^2}{2}\Big)\Big] = -m^2(L_1 + 2mM_1) \\
&a_0a_1\Big[3 + 2m + \frac{a_2}{a_0}(5 + 2m) - 3m^2\frac{a_{-2}}{a_0} + 3m^2\frac{a_{-4}}{a_0}\Big] \\
&+ a_0a_3\Big[\frac{a_2}{a_0}(7 + 2m) + \frac{a_4}{a_0}(9 + 2m)\Big] \\
&+ a_0a_{-1}\Big[1 + 2m - 3m^2 + \frac{a_{-2}}{a_0}\Big(-1 + 2m + 3m^2\Big)\Big] \\
&+ a_0a_{-3}\Big[3m^2 - 3m^2\frac{a_2}{a_0} + (-3 + 2m)\frac{a_{-2}}{a_0} + (-5 + 2m)\frac{a_{-4}}{a_0}\Big] = m^2M_1.
\end{aligned}\right\}\ (6)$$

The equations (6) will be used for $a_1$ and $a_{-1}$ instead of those obtained from the general form (5).

## IV.

### *The Determination of the Parts of $a_i$ depending on the First Power of the Ratio of the Mean Distances.*

The coefficients $a_i$ will now be obtained to the order $m^5.a_0/a'$. To obtain them to this order we have

$$
\begin{array}{ll}
(u^3)_1 = 3a_0^2 a_{-2}, & (u^2 s)_1 = a_0^3 + 2a_0(a_2^2 + a_{-2}^2 + a_2 a_{-2}), \\
(s^3)_1 = 3a_0^2 a_{-4} + 3a_{-2}^2 a_0, & (us^2)_1 = a_0^2(2a_{-2} + a_2), \\
(u^3)_3 = (s^3)_{-3} = a_0^3, & (u^2 s)_3 = (us^2)_{-3} = a_0^2(2a_2 + a_{-2}), \\
(u^3)_5 = (s^3)_{-5} = 3a_0^2 a_2, &
\end{array}
$$

the rest of the coefficients of $\zeta$ in $u^3$, $s^3$, $u^2s$, $s^2u$ being zero to the order taken. Whence we have

$$L_1 = \frac{a_0}{a'}\left[\frac{15}{2}(a_0a_{-2}+a_0a_{-4}+a_{-2}^2)+\frac{3}{2}(a_0^2+2a_2^2+2a_{-2}^2+2a_2a_{-2}+2a_0a_{-2}+a_0a_2)\right],$$

$$M_1 = \frac{a_0}{a'}\left[\frac{45}{8}(a_0a_{-2}-a_0a_{-4}-a_{-2}^2)+\frac{3}{8}(a_0^2+2a_2^2+2a_{-2}^2+2a_2a_{-2}-2a_0a_{-2}-a_0a_2)\right]$$

for substitution in the equations (6). Also the right-hand side of equation (5) becomes when

$$
\begin{array}{ll}
i = \phantom{-}3, & \dfrac{a_0}{a'}[A_3a_0^2 + B_3a_0(2a_2 + a_{-2})], \\[2ex]
i = -3, & \dfrac{a_0}{a'}[A'_{-3}a_0^2 + B'_{-3}a_0(2a_2 + a_{-2})], \\[2ex]
i = \phantom{-}5, & \dfrac{a_0}{a'}A_5.3a_0a_2, \\[2ex]
i = -5, & \dfrac{a_0}{a'}A'_{-5}.3a_0a_2,
\end{array}
$$

whence, writing out equations (5) for $a_3$, $a_{-3}$, $a_5$, $a_{-5}$ with a sufficient number of terms to obtain these quantities as far as the order $m^5.a_0/a'$, we have

$$
\left.\begin{array}{rl}
a_0a_3 = & [3,\,1]\,a_1a_{-2} + [3,\,2]\,a_{-1}a_2 + [3]\,2a_1a_0 + \dfrac{a_0}{a'}[A_3a_0^2 + B_3a_0(2a_2 + a_{-2})] \\
& \qquad + [3,\,4]\,a_1a_4 + [3,\,-1]\,a_{-1}a_{-4} + [3]\,2a_2a_{-1}, \\
a_0a_{-3} = & [-3,\,-2]\,a_1a_{-2} + [-3,\,-1]\,a_{-1}a_2 + (-3)\,2a_1a_0 \\
& \qquad + \dfrac{a_0}{a'}[A'_{-3}a_0^2 + B'_{-3}a_0(2a_2 + a_{-2})] \\
& \qquad + [-3,\,1]\,a_1a_4 + [-3,\,-4]\,a_{-1}a_{-4} + (-3)\,2a_2a_{-1},
\end{array}\right\}\quad(7)
$$

$$\left.\begin{aligned} a_0a_5 &= [5,\,1]\,a_1a_{-4} + [5,\,4]\,a_4a_{-1} + [5,\,2]\,a_2a_{-3} \\ &\qquad + [5,\,3]\,a_3a_{-2} + [5](2a_3a_0 + 2a_2a_1) + \frac{a_0}{a'}A_5.3a_0a_2, \\ a_0a_{-5} &= [-5,\,-4]\,a_1a_{-4} + [-5,\,-1]\,a_4a_{-1} + [-5,\,-3]\,a_2a_{-3} \\ &\qquad + [-5,\,-2]\,a_3a_{-2} + (-5)(2a_3a_0 + 2a_2a_1) + \frac{a_0}{a'}A'_{-5}.3a_0a_2. \end{aligned}\right\}\quad (8)$$

The usual method is to solve the equations (6) for a first approximation of $a_1$, $a_{-1}$ by neglecting $a_3$, $a_{-3}$, obtain $a_3$, $a_{-3}$ from (7) with these values of $a_1$, $a_{-1}$ and proceed then for a second approximation of $a_1$, $a_{-1}$. I prefer to do this in one process as follows.

Since $a_2$, $a_{-2}$, $a_4$, $a_{-4}$ are known expansions of $m$ in series, we can to the order given, write the equations for $a_3$, $a_{-3}$ in the form

$$\left.\begin{aligned} a_3 &= \alpha a_1 + \beta a_{-1} + \gamma\frac{a_0}{a'}a_0, \\ a_{-3} &= \alpha' a_1 + \beta' a_{-1} + \gamma'\frac{a_0}{a'}a_0, \end{aligned}\right\}\quad (9)$$

where $\alpha$, $\beta$, $\gamma$, $\alpha'$, $\beta'$, $\gamma'$ are known functions of $m$. Substituting these values of $a_3$, $a_{-3}$ in equations (6) for $a_1$, $a_{-1}$, we obtain linear equations of the form

$$\left.\begin{aligned} a_1.\varkappa + a_{-1}.\lambda &= \mu.\frac{a_0}{a'}a_0, \\ a_1.\varkappa' + a_{-1}.\lambda' &= \mu'.\frac{a_0}{a'}a_0, \end{aligned}\right\}\quad (10)$$

where $\varkappa$, $\lambda$, $\mu$, $\varkappa'$, $\lambda'$, $\mu'$ are known functions of $m$. This method of procedure has the advantage of giving us the whole value of the functions of $m$ used when dealing numerically with the equations; while in finding their expansions algebraically, we lose much less than by the usual method. Also, if we wish to proceed to a higher approximation, by using the equations always in the forms (9) and (10), we shall merely get terms in (9) independent of $a_1$, $a_{-1}$, and consequently in (10) terms added only to the right-hand sides of the equations. Then the coefficients of $a_1$, $a_{-1}$ in (10) will remain unchanged, and the small changes in $\mu$, $\mu'$ can be easily obtained. This will not be found necessary here, as the results obtained up to the order indicated above are sufficiently accurate.

Expressed in the form (9), equations (7) become

$$\left.\begin{aligned}
a_3 &= m^2\left(\frac{51}{2^7}+\frac{3371}{3^2.2^9}m+\frac{13451}{3^3.2^{10}}m^2\right)a_1 - m^2\left(\frac{15}{2^7}+\frac{167}{2^9}m+\frac{607}{3.2^9}m^2\right)a_{-1}\\
&\quad + m^2\left(\frac{5}{2^7}+\frac{45}{2^9}m+\frac{545}{3.2^{11}}m^2-\frac{781}{3^2.2^{12}}m^3\right)\frac{a_0}{a'}a_0,\\
a_{-3} &= m^2\left(-\frac{25}{2^7}+\frac{103}{3^2.2^9}m+\frac{2659}{3^3.2^{10}}m^2\right)a_1 - m^2\left(\frac{3}{2^7}+\frac{19}{2^9}m+\frac{215}{3.2^9}m^2\right)a_{-1}\\
&\quad - m_2\left(\frac{55}{2^7}+\frac{175}{2^9}m-\frac{229}{3.2^{11}}m^2-\frac{8455}{3^2.2^{12}}m^3\right)\frac{a_0}{a'}a_0,
\end{aligned}\right\}(11)$$

and therefore from equation (6) we can obtain

$$\left.\begin{aligned}
&a_1\left(3+\frac{29}{16}m^2+\frac{7}{2}m^3-\frac{1163}{3.2^{10}}m^4-\frac{209893}{3^2.2^{12}}m^5\right)\\
&\qquad + a_{-1}\left(1+\frac{61}{16}m^2+\frac{13}{3}m^3-\frac{68563}{3^2.2^{10}}m^4-\frac{338477}{3^3.2^{12}}m^5\right)\\
&= -\frac{a_0}{a'}.\,a_0\left(\frac{3}{2}m^2+\frac{3}{4}m^3-\frac{12795}{2^{10}}m^4-\frac{96643}{2^{12}}m^5-\frac{752323}{3.2^{15}}m^6\right),\\
&a_1\left(3+2m+\frac{15}{16}m^2+\frac{23}{8}m^3+\frac{20645}{3.2^{10}}m^4+\frac{362467}{3^2.2^{12}}m^5\right)\\
&\qquad + a_{-1}\left(1+2m-\frac{29}{16}m^2-\frac{17}{24}m^3-\frac{55379}{3^2.2^{10}}m^4-\frac{891109}{3^3.2^{12}}m^5\right)\\
&= \frac{a_0}{a'}.\,a_0\left(\frac{3}{8}m^2-\frac{3165}{2^{10}}m^4-\frac{21333}{2^{12}}m^5-\frac{1300401}{3.2^{15}}m^6\right).
\end{aligned}\right\}(11a)$$

Subtract the first of these equations from the second, the resulting equation is then divisible by $m$; also, multiply the second equation by 3 and subtract from the first. We have then two equations from which $a_1$, $a_{-1}$ must be found. Performing these processes, we get

$$\left.\begin{aligned}
\frac{a_1}{a_0} &= \frac{a_0}{a'}\cdot\frac{1}{m\tau}\left[-\frac{15}{32}m^2-\frac{15}{16}m^3+\frac{123}{32}m^4+\frac{13975}{2^{10}}m^5+\frac{1709047}{3.2^{15}}m^6\right],\\
\frac{a_{-1}}{a_0} &= \frac{a_0}{a'}\cdot\frac{1}{m\tau}\left[+\frac{45}{32}m^2+\frac{21}{16}m^3-\frac{2763}{2^8}m^4-\frac{13449}{2^9}m^5-\frac{2979511}{3.2^{15}}m^6\right],
\end{aligned}\right\}(12)$$

where $m\tau$, the common divisor, is given by

$$m\tau = m - 4m^2 - \frac{37}{8}m^3 - \frac{17}{6}m^4 - \frac{89963}{3^2.2^{10}}m^5.$$

We can of course divide out by $m$. It has been kept in here to show in what manner a transformation of the form

$$m = \frac{m'}{1 + am'},$$

if indicated by theory, must be made.* We should get different expressions for the coefficients $a_1$ and $a_{-1}$, kept still in the *form* (12) according as the factor $m$ had been divided out or not.

Using the values (12), after dividing out by the factor $m$, since it can now be any time replaced if necessary, equations (11) give

$$\frac{a_3}{a_0} = m^2 . \frac{a_0}{a'} \left[ \frac{5}{2^7} + \frac{45}{2^9} m + \frac{545}{3 . 2^{11}} m^2 - \frac{781}{3^2 . 2^{12}} m^3 - \frac{m}{\tau} \left( \frac{45}{2^7} + \frac{8165}{3 . 2^{11}} m - \frac{66251}{3^2 . 2^{13}} m^2 \right) \right],$$

$$\frac{a_{-3}}{a_0} = m^2 . \frac{a_0}{a'} \left[ - \frac{55}{2^7} - \frac{175}{2^9} m + \frac{229}{3 . 2^{11}} m^2 + \frac{8455}{3^2 . 2^{12}} m^3 + \frac{m}{\tau} \left( \frac{15}{2^8} + \frac{551}{3 . 2^{11}} m - \frac{7459}{3^2 . 2^{10}} m^2 \right) \right],$$

and thence from equations (8),

$$\frac{a_5}{a_0} = \frac{a_0}{a'} m^4 \left[ \frac{105}{2^{11}} + \frac{797}{2^{12}} m - \frac{2655}{2^{13}} . \frac{m}{\tau} \right],$$

$$\frac{a_{-5}}{a_0} = \frac{a_0}{a'} m^4 \left[ - \frac{65}{2^{11}} - \frac{523}{3 . 2^{11}} m - \frac{75}{2^{13}} . \frac{m}{\tau} \right].$$

This method has the advantage also of giving numerical results along with the algebraical expansions, and thus we can obtain some idea of the errors which are produced by the portions of the series neglected when we expand the various functions used, in powers of $m$. In working out the numerical results by this method, it appeared that the principal parts of the errors in $a_1$, $a_{-1}$ were due to the neglect of the higher powers of $m$ in solving the linear equations for $a_1$ and $a_{-1}$, and that even this part was not very great. For example, the value of $\tau$ found was

$$1 - 4m - \frac{37}{8} m^2 - \frac{17}{6} m^3 - \frac{89963}{3^2 . 2^{10}} m^4,$$

giving $\tau = .6443757$.

The value found by the numerical process to the same order is

$$\tau = .6444540,$$

a difference of $.0000783$.

*See Monthly Not. R. A. S., Vol. LII, No. 2.

The term
$$-\frac{89963}{3^2.2^{10}}m^4 = -.0004171,$$
so that the neglected portion is less than one-fifth of the last term in $\tau$ calculated.

The expressions given above, when transformed, agree with those given by Delaunay as far as the order $m^4.a_0/a'$. The multiplier of $m^5.a_0/a'$ differs in all these coefficients by a small amount, causing, however, only a small difference in the numerical values of the coefficients of a few ten-thousandths of a second.

## V.

### *The Determination of the Portions of the Coefficients depending on Powers of $1/a'$ higher than the first.*

If now we wish to obtain the portions of the coefficients depending on $(a_0/a')^2$ and $(a_0/a')^2$, very little extra labor is necessary. In the coefficients with even suffixes only even powers, and in those with odd suffixes, only odd powers of this ratio occur. Let $\delta a_r$ denote the portion to be added to $a_r$ depending on the square or cube of this ratio according as $r$ is even or odd. The even coefficients will be obtained below as far as the order $m^3(a_0/a')^2$, and the odd ones as far as the order $m^2(a_0/a')^3$. As $\delta a_0$ is of the order $m^2(a_0/a')^2$, we have to the required degree of accuracy from equation (5), after obtaining the necessary terms in $L$, $M$,

$$\left.\begin{aligned}
a_0\delta a_2 &= [2,\,1]\,a_1a_{-1} + [2,\,3]\,a_3a_1 + [2,\,-1]\,a_{-1}a_{-3} \\
&\qquad + \frac{a_0}{a'}[3A_2a_0a_{-1} + B_2a_0(2a_1 + a_{-1})] + \left(\frac{a_0}{a'}\right)^2 D_2a_0^2, \\
a_0\delta a_{-2} &= [-2,\,-1]\,a_1a_{-1} + [-2,\,1]\,a_3a_1 + [-2,\,-3]\,a_{-1}a_{-3} \\
&\qquad + \frac{a_0}{a'}[3A'_{-2}a_0a_{-1} + B'_{-2}a_0(2a_1 + a_{-1})] + \left(\frac{a_0}{a'}\right)^2 D'_{-2}a_0^2, \\
a_0\delta a_4 &= [4,\,3]\,a_3a_{-1} + [4,\,1]\,a_1a_{-3} + \frac{a_0}{a'}.\,3A_4a_0a_1 + \left(\frac{a_0}{a'}\right)^2 C_4a_0^2, \\
a_0\delta a_{-4} &= [-4,\,-1]\,a_3a_{-1} + [-4,\,-3]\,a_1a_{-3} + \frac{a_0}{a'}.\,3A'_{-4}a_0a_1 + \left(\frac{a_0}{a'}\right)^2 C'_{-4}a_0^2, \\
a_0\delta a_3 &= \left(\frac{a_0}{a'}\right)^3 G_3a_0^2, \qquad\qquad a_0\delta a_{-3} = \left(\frac{a_0}{a'}\right)^3 G'_{-3}a_0^2, \\
a_0\delta a_5 &= \left(\frac{a_0}{a'}\right)^3 F_5a_0^2, \qquad\qquad a_0\delta a_{-5} = \left(\frac{a_0}{a'}\right)^3 F'_{-5}a_0^2,
\end{aligned}\right\} \quad (13)$$

For $a_1$ and $a_{-1}$ we must use the equations (6). Differentiating these and remembering the orders to which our quantities are carried, we get

$$a_0\delta a_1 P + a_0\delta a_{-1}P' + 7a_1\delta a_2 + a_{-1}\delta a_{-2} = -m^2(\delta L_1 + 2m\delta M_1),$$
$$a_0\delta a_1 Q + a_0\delta a_{-1}Q' + 5a_1\delta a_2 - a_{-1}\delta a_{-2} = m^2\delta M_1,$$

where $P$, $Q$, $P'$, $Q'$ are the same functions of $m$ which we had previously as multipliers of $a_1$ and $a_{-1}$ in their respective equations. Treat these in the same manner as we did the corresponding ones for $a_1$, $a_{-1}$, i. e. subtract the first equation from the second and also multiply the second equation by 3 and add to the first, we get

$$a_0\delta a_1(Q-P)+a_0\delta a_{-1}(Q'-P') = m^2[\delta L_1 + \delta M_1(1+2m)] + 2a_1\delta a_2 + 2a_{-1}\delta a_{-2},$$
$$a_0\delta a_1(3Q+P)+a_0\delta a_{-1}(3Q'+P') = m^2[-\delta L_1 + \delta M_1(3-2m)] - 22a_1\delta a_2 + 2a_{-1}\delta a_{-2},$$

corresponding to the equations (11$a$), the coefficients on the left-hand side being the same. Also

$$\delta L_1 = a_0\left(\frac{a_0}{a'}\right)^2\left[\frac{25}{16}(3a_{-1}+a_1) + \frac{45}{32}2(a_{-1}+a_1) + \frac{45}{32}a_0\right],$$
$$\delta M_1 = a_0\left(\frac{a_0}{a'}\right)^2\left[\frac{5}{8}(3a_{-1}+a_1) + \frac{15}{64}a_0\right].$$

Working out these and equations (13), and retaining the factor $\frac{1}{\tau}$ wherever it occurs, we obtain the following series of values:

$$\frac{\delta a_2}{a_0} = m^2\left(\frac{a_0}{a'}\right)^2\left[\frac{5}{2^6} + \frac{5}{2^4}m + \frac{45.67}{2^{12}}\cdot\frac{m}{\tau} + \frac{45}{2^{11}}(15+49m)\frac{1}{\tau^2}\right],$$
$$\frac{\delta a_{-2}}{a_0} = m^2\left(\frac{a_0}{a'}\right)^2\left[-\frac{45}{2^6} - \frac{5}{3.2^3}m - \frac{15.1253}{2^{12}}\cdot\frac{m}{\tau} + \frac{45}{2^{11}}(5+3m)\frac{1}{\tau^2}\right],$$
$$\frac{\delta a_4}{a_0} = m^2\left(\frac{a_0}{a'}\right)^2\left[\frac{7}{2^9} + \frac{7}{3.5.2^4}m - \frac{15.25}{2^{12}}\cdot\frac{m}{\tau}\right],$$
$$\frac{\delta a_{-4}}{a_0} = m^2\left(\frac{a_0}{a'}\right)^2\left[-\frac{7.17}{2^9} - \frac{21}{5.2^5}m + \frac{15.45}{2^{12}}\cdot\frac{m}{\tau}\right],$$
$$\frac{\delta a_1}{a_0} = \left(\frac{a_0}{a'}\right)^3\left[-\frac{105}{2^8}m - \frac{210}{2^8}m^2 - \frac{1125}{2^9}\cdot\frac{m^2}{\tau}\right]\frac{1}{\tau},$$
$$\frac{\delta a_{-1}}{a_0} = \left(\frac{a_0}{a'}\right)^3\left[\frac{315}{2^8}m + \frac{270}{2^8}m^2 + \frac{3375}{2^9}\cdot\frac{m^2}{\tau}\right]\frac{1}{\tau},$$
$$\frac{\delta a_3}{a_0} = m^2\cdot\left(\frac{a_0}{a'}\right)^3\cdot\frac{35}{3.2^{11}},$$
$$\frac{\delta a_{-3}}{a_0} = m^2\cdot\left(\frac{a_0}{a'}\right)^3\left(-\frac{35.43}{3.2^{11}}\right),$$
$$\frac{\delta a_5}{a_0} = m^2\cdot\left(\frac{a_0}{a'}\right)^3\cdot\frac{21}{5.2^{11}},$$
$$\frac{\delta a_{-5}}{a_0} = m^2\cdot\left(\frac{a_0}{a'}\right)^3\left(-\frac{1533}{5.2^{11}}\right).$$

When these are compared with Delaunay's expressions they all agree as far as the order to which he has carried them.

The results obtained by using numerical values from the outset are:

$$\frac{a_1}{a_0} = -.0641700 \frac{a_0}{a'} - .086 \left(\frac{a_0}{a'}\right)^3,$$

$$\frac{a_{-1}}{a_0} = +.1789909 \frac{a_0}{a'} + .240 \left(\frac{a_0}{a'}\right)^3,$$

$$\frac{a_3}{a_0} = -.0000589 \frac{a_0}{a'} + .0001 \left(\frac{a_0}{a'}\right)^3,$$

$$\frac{a_{-3}}{a_0} = -.0029370 \frac{a_0}{a'} - .0016 \left(\frac{\cdot a_0}{a'}\right)^3,$$

$$\frac{a^5}{a_0} = +.00000047 \frac{a_0}{a'} + .0000 \left(\frac{a_0}{a'}\right)^3,$$

$$\frac{a_{-5}}{a_0} = -.00000184 \frac{a_0}{a'} - .0010 \left(\frac{a_0}{a'}\right)^3,$$

$$\frac{\delta a_2}{a_0} = +.007284 \left(\frac{a_0}{a'}\right)^2, \qquad \frac{\delta a_{-2}}{a_0} = -.007432 \left(\frac{a_0}{a'}\right)^2,$$

$$\frac{\delta a_4}{a_0} = +.000060 \left(\frac{a_0}{a'}\right)^2, \qquad \frac{\delta a_{-4}}{a_0} = -.001428 \left(\frac{a_0}{a'}\right)^2.$$

The quantity $\delta a_0$ must now be obtained. For this purpose I use the equation

$$\left(D^2 + 2mD - \frac{\varkappa}{(us)^{\frac{3}{2}}}\right) u = -\frac{3}{2} m^2 (u+s) - m^2 \cdot \frac{2}{n'^2} \cdot \frac{d\Omega_1}{ds}.$$

Substituting in this

$$u = \Sigma a_{i-1} \zeta^i, \quad s = \Sigma a_{-i-1} \zeta^i,$$

and taking out the coefficient of $\zeta$ we get with the former notation,

$$\varkappa \left(\frac{u}{(us)^{\frac{3}{2}}}\right)_1 = \left(1 + 2m + \frac{3}{2} m^2\right) a_0 + \frac{3}{2} m^2 a_{-2} + \frac{m^2}{a'} \left[\frac{15}{8} (s^2)_1 + \frac{3}{8} (u^2 + 2us)_1\right]$$
$$+ \frac{m^2}{a'^2} \left[\frac{35}{16} (s^3)_1 + \frac{5}{16} (u^3)_1 + \frac{15}{16} (us^2)_1 + \frac{9}{16} (u^2 s)_1\right] + \dots.$$

When the parallactic terms are neglected Hill finds that

$$a_0 = a \left[\frac{J(1+m)^2}{H}\right]^{\frac{1}{3}},$$

where

$$\mu = n^2 a^3, \qquad J = a_0^2 \left(\frac{u}{(us)^{\frac{3}{2}}}\right)_1,$$

and $a_0 H$ is the value of the right-hand side of the equation. Let $\delta a_0$, $\delta J$, $\delta H$ be the corresponding quantities due to the parallactic terms. Then since

$$J + \delta J, \quad H + \delta H$$

are defined in the same way as $J$, $H$, we have

$$a_0 + \delta a_0 = a\left[\frac{(J+\delta J)(1+m)^2}{H+\delta H}\right]^{\frac{1}{3}}.$$

Neglecting powers of $\frac{a_0}{a'}$ above the second, we get

$$\frac{\delta a_0}{a_0} = \frac{1}{3}\cdot\frac{\delta J}{J} - \frac{1}{3}\cdot\frac{\delta H}{H}.$$

We shall obtain $\delta a_0$ to the order $m^3.(a_0/a')^2$. For this

$$\delta H = m^2.\frac{a_0}{a'}\left[\frac{3}{8}\cdot 2\frac{a_{-1}}{a_0} + \frac{3}{4}\frac{(a_1+a_{-1})}{a_0}\right] + m^2.\frac{a_0^2}{a'^2}\cdot\frac{9}{16}$$

$$= \left[\frac{9}{16}m^2 + \frac{225}{2^7}\cdot\frac{m^3}{\tau}\right]\left(\frac{a_0}{a'}\right)^2.$$

$J+\delta J$ is the coefficient of $\zeta$ in $\frac{a_0^2 u}{(us)^{\frac{3}{2}}}$ or of $\zeta^0$ in $\frac{(\Sigma a_i\zeta^i)\,a_0^2}{(\Sigma a_j a_{-i+j}\zeta^i)^{\frac{3}{2}}}$; whence we get

$$\delta J = -\frac{3}{2}\cdot\frac{a_1^2+a_{-1}^2}{a_0^2} + \frac{9}{4}\cdot\frac{(a_1+a_{-1})^2}{a_0^2}$$

$$= -\frac{m^2}{\tau^2}\cdot\frac{675}{2^9}(1+4m)\left(\frac{a_0}{a'}\right)^2.$$

Also

$$H = 1 + 2m + \frac{3}{2}m^2 + \frac{3}{2}m^2\frac{a_{-2}}{a_0},$$

$$J = 1 + \frac{21}{2^8}m^4 + \ldots.,$$

$$a_0 = a\left(1 - \frac{m^2}{6} - \ldots.\right),$$

we therefore have

$$\frac{\delta a_0}{a_0} = -\left(\frac{a}{a'}\right)^2 m^2\left\{\frac{225}{512}\cdot\frac{1+4m}{\tau^2} + \frac{75}{128}\cdot\frac{m}{\tau} + \frac{3}{16}(1-2m)\right\},$$

and numerically,

$$\frac{\delta a_0}{a_0} = -\left(\frac{a_0}{a'}\right)^2.00965.$$

## VI.

### *Transformation to Polar Coordinates.*

The inequalities will now be expressed in polar coordinates. For this purpose we have, if $V$ be the true longitude of the moon,

$$\begin{aligned} r\cos(V - nt) &= \frac{1}{2}\Sigma a_i(\zeta^i + \zeta^{-i}) \\ &= a_0\left\{1 + \frac{1}{2}\Sigma\frac{a_i}{a_0}(\zeta^i + \zeta^{-i})\right\} \\ &= a_0\left(1 + \frac{1}{2}\Sigma A_i\zeta^i\right), \\ r\sin(V - nt) &= \frac{1}{2\sqrt{-1}}\Sigma a_i(\zeta^i - \zeta^{-i}) \\ &= a_0\frac{1}{2\sqrt{-1}}\Sigma B_i\zeta^i, \end{aligned}$$

where $A_i . a_0 = a_i + a_{-i}$ and $B_i . a_0 = b_i - b_{-i}$, and consequently $A_i = A_{-i}$, $B_i = -B_{-i}$. And since

$$\tan\theta = \theta - \frac{\theta^3}{3} + \ldots\,,$$

$$V - nt = \frac{1}{2\sqrt{-1}}[\Sigma B_i\zeta^i]\left[1 - \frac{1}{2}\Sigma A_i\zeta^i + \frac{1}{4}(\Sigma A_i\zeta^i)^2 - \ldots\right] + \frac{1}{24\sqrt{-1}}[\Sigma B_i\zeta^i]^3[1 - \ldots]^3 + \ldots$$

The coefficient of $\sin D$ or $(\zeta^1 - \zeta^{-1})/2\sqrt{-1}$ is therefore

$$B_1 + \frac{1}{2}(B_1A_2 - A_1B_2) + \frac{1}{2}(B_2A_3 - A_2B_3) + \frac{1}{2}B_1(A_2^2 - B_2^2) + \ldots\,,$$

or $$\frac{a_1 - a_{-1}}{a_0} + \frac{a_1a_{-2} - a_2a_{-1} + a_2a_{-3} - a_3a_{-2}}{a_0^2} + \frac{2(a_1 - a_{-1})\,a_2a_{-2}}{a_0^3} + \ldots\,,$$

and similarly for the other coefficients. Performing these operations we get the following expressions, omitting those terms dependent on $m$ only.

In longitude,

$$\left[-\underbrace{\frac{\frac{15}{8}m + \frac{9}{4}m^2 - \frac{1951}{128}m^3 - \frac{41585}{1024}m^4 - \frac{2096751}{49152}m_5}{1 - 4m - \frac{37}{8}m^2 - \frac{17}{6}m^3 - \frac{89963}{9216}m^4 (= \tau)}}_{-128''.0545} - \underset{-0''.0008}{\frac{35}{1024}m^4} - \underset{-0''.0002}{\frac{1345}{12288}m^5}\right.$$

$$\left. - \frac{a^2}{a'^2}\left\{\underset{-0''.0006}{\frac{105}{64}\cdot\frac{m}{\tau}} + \underset{-0''.0001}{\frac{15}{8}\,\frac{m^2}{\tau}} + \underset{-0''.0004}{\frac{1125}{128}\cdot\frac{m^2}{\tau^2}}\right\}\right]\frac{a}{a'}\sin D$$

$$+\left[\underset{0''.0078}{\frac{5m^2}{32}(5+8m)}+\underset{+0''.0276}{\frac{225}{1024}m^2\frac{(5+11m)}{\tau^2}}+\underset{+0''0060}{\frac{15.747}{2048}\cdot\frac{m^3}{\tau}}\right]\frac{a^2}{a'^2}\sin 2D$$

$$+\left[\underset{1''.6172}{\frac{15}{32}m^2}+\underset{+0''.1199}{\frac{55}{128}m^3}-\underset{-0''.0006}{\frac{41}{1536}m^4}-\underset{-0''.0005}{\frac{2309}{9216}m^5}-\underset{-0''.9872}{\frac{1}{\tau}\left(\frac{255}{128}m^3+\frac{7543}{1536}m^4-\frac{134525}{8192}m^5\right)}\right.$$

$$\left.+\underset{+0''.0000}{\frac{a^2}{a'^2}\cdot\frac{385}{1536}m^2}\right]\frac{a}{a'}\sin 3D$$

$$+\left[\underset{0''.0021}{\frac{63}{256}m^2}+\underset{+0''.0001}{\frac{77}{480}m^3}-\underset{-0''.0008}{\frac{1725}{2048}\cdot\frac{m^3}{\tau}}\right]\frac{a^2}{a'^2}\sin 4D$$

$$+\left[\underset{0''.0132}{\frac{75}{128}m^4}+\underset{+0''.0025}{\frac{2797}{2048}m^5}-\underset{-0''.0064}{\frac{18415}{8192}\cdot\frac{m^5}{\tau}}+\underset{+0''.0000}{\frac{a^2}{a'^2}\cdot\frac{777}{5120}m^2}\right]\frac{a}{a'}\sin 5D.$$

In parallax,

$$\frac{1}{a}\left[-\underset{-1''.0100}{\left(\frac{15}{16}m+\frac{3}{8}m^2-\frac{1869}{256}\right)\frac{1}{\tau}}\right]\frac{a}{a'}\cos D$$

$$+\frac{1}{a}\left[\underset{0''.0223}{\frac{25}{64}m^2}+\underset{+0''.0012}{\frac{65}{256}m^3}-\underset{-0''.0139}{\frac{459}{256}\cdot\frac{m^3}{\tau}}\right]\frac{1}{a'}\cos 3D.$$

None of the other parts produce in parallax, coefficients so great as one thousandth of a second.

The numerical values of these coefficients are obtained by using

$$m = .0808489338,$$

$$\frac{a}{a'} = .00255878.$$

If we use the numerical value of $m$ from the outset, we obtain the more accurate series of coefficients:

In longitude,

$$-128''.070\sin D+0''.039\sin 2D+0''.750\sin 3D+0''.001\sin 4D+0''.008\sin 5D,$$

and in parallax

$$-1''.001\cos D+0''.008\cos 3D.$$

For the discussion and comparison of these results with those obtained by Delaunay, see Monthly Notices of Royal Astron. Soc., Vol. LII, No. 2.

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